Université Nice-Sophia Antipolis
SMEFA302 - ECUE Probabilistic computational methods, 2024-2025
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Exercices (3)

Let $(Z_n)_{n\geq 1}$ be an i.i.d. sequence of $\mathcal{N}(0,1)$-distributed random variables defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$.

(1) Compute for every real numbers $a,b>0$, the quantity $\mathbb{E}((1+bZ_1)e^{aZ_1})$.

$$\begin{aligned}
\mathbb{E}((1+bZ_1)e^{aZ_1}) &= \int_{\mathbb{R}}(1+bz)e^{az}\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz \\
&= \int_{\mathbb{R}}(1+ba+b(z-a))\frac{e^{-\frac{1}{2}(z-a)^2}e^{a^2/2}}{\sqrt{2\pi}}dz \\
\text{(integral of odd integrable function is zero)} &= (1+ba)e^{a^2/2}+0
\end{aligned}$$

(2) Compute, for every integer $n\geq 1$,

$$\left\| e^{a(Z_1+\dots+Z_n)-na^2/2}-\prod_{k=1}^n(1+aZ_k)\right\|_2$$

(remember : for all real r.v. X, $\|X\|_2=\sqrt{\mathbb{E}(X^2)}$).

$$\left\| e^{a(Z_1+\dots+Z_n)-na^2/2}-\prod_{k=1}^n(1+aZ_k)\right\|_2^2 = \mathbb{E}\left(\left(e^{a(Z_1+\dots+Z_n)-na^2/2}-\prod_{k=1}^n(1+aZ_k)\right)^2\right)$$

$$= \mathbb{E}\left(e^{2a(Z_1+\dots+Z_n)-na^2}\right)+\mathbb{E}\left(\prod_{k=1}^n(1+aZ_k)^2\right)-2\mathbb{E}\left(e^{a(Z_1+\dots+Z_n)-na^2/2}\prod_{k=1}^n(1+aZ_k)\right)$$

$(Z_1,\dots,Z_n$ independant)

$$= \left(\mathbb{E}(e^{2aZ_1-a^2})\right)^n+\left(\mathbb{E}((1+aZ_1)^2)\right)^n-2\left(\mathbb{E}(e^{aZ_1-a^2/2}(1+aZ_1))\right)^n$$

(we use Question 1)

$$= \left(e^{\frac{4a^2}{2}-a^2}\right)^n+\left(1+a^2\right)^n-2\left((1+a^2)\right)^n$$

$$= e^{na^2}-(1+a^2)^n$$

so

$$\left\| e^{a(Z_1+\dots+Z_n)-na^2/2}-\prod_{k=1}^n(1+aZ_k)\right\|_2 = \sqrt{e^{na^2}-(1+a^2)^n}\,.$$

(3) Let $\sigma>0$. Show the existence of real constants c_1, c_2 such that

$$\left\| e^{\sigma(Z_1+\dots+Z_n)/\sqrt{n}-\sigma^2/2}-\prod_{k=1}^n\left(1+\frac{\sigma Z_k}{\sqrt{n}}\right)\right\|_2 = \frac{\sigma e^{\sigma^2/2}}{\sqrt{2n}}\left(1-\frac{\sigma^2(c_1\sigma^2+c_2)}{n}+O(1/n^2)\right).$$

Use the previous question:

$$\left\| e^{\sigma(Z_1+\cdots+Z_n)/\sqrt{n}-\sigma^2/2} - \prod_{k=1}^{n}\left(1+\frac{\sigma Z_k}{\sqrt{n}}\right)\right\|_2 = \left(e^{\sigma^2} - \left(1+\frac{\sigma^2}{n}\right)^n\right)^{1/2}$$

$$= \left(e^{\sigma^2} - \exp\left(n\ln\left(1+\frac{\sigma^2}{n}\right)\right)\right)^{1/2}$$

$$= \left(e^{\sigma^2} - \exp\left(n\left(\frac{\sigma^2}{n} - \frac{1}{2}\frac{\sigma^4}{n^2} + \frac{1}{3}\frac{\sigma^6}{n^3} + O(1/n^4)\right)\right)\right)^{1/2}$$

$$= e^{\sigma^2/2}\left(1 - \exp\left(-\frac{1}{2}\frac{\sigma^2}{n} + \frac{1}{3}\frac{\sigma^4}{n^2} + O(1/n^3)\right)\right)^{1/2}$$

$$= e^{\sigma^2/2}\left(\frac{1}{2}\frac{\sigma^2}{n} - \frac{1}{3}\frac{\sigma^4}{n^2} - \frac{1}{2!}\frac{\sigma^4}{4n^2} + O(1/n^3)\right)^{1/2}$$

$$= \frac{\sigma e^{\sigma^2/2}}{\sqrt{2n}}\left(1 - \frac{2\sigma^2}{3n} - \frac{\sigma^4}{4n} + O(1/n^2)\right)^{1/2}$$

$$= \frac{\sigma e^{\sigma^2/2}}{\sqrt{2n}}\left(1 - \frac{1}{2}\left(\frac{2\sigma^2}{3n} + \frac{\sigma^4}{4n}\right) + O(1/n^2)\right)$$

(4) We are interested in the equation

$$dX_t = \sigma X_t dW_t \tag{0.1}$$

(where $(W_t)_{t\geq 0}$ is a standard Brownian motion). Show that the (strong) solution of this equation is

$$X_t = X_0 \exp(\sigma W_t - \sigma^2 t/2)\,.$$

The equation is of the form $dX_t = b(X_t)dW_t$ with a b which is Lipschitz in x so there is a unique strong solution. Apply Itō's formula:

$$e^{\sigma W_t - \sigma^2 t/2} = e^{\sigma W_0} + \int_0^t \sigma e^{\sigma W_s - \sigma^2 s/2} dW_s + \int_0^t -\frac{\sigma^2}{2} e^{\sigma W_s - \sigma^2 s/2} ds + \frac{1}{2}\int_0^s \sigma^2 e^{\sigma W_s - s^2/2} ds$$

$$= e^{\sigma W_0} + \int_0^t \sigma e^{\sigma W_s - \sigma^2 s/2} dW_s\,.$$

So we have our solution.

(5) We suppose that the increments of (W_t) on $[0,1]$ are such that $W_{\frac{k}{n}} - W_{\frac{k-1}{n}} = Z_k/\sqrt{n}$, for all $k \in \{1,2,\dots,n\}$. Show that the Euler scheme of order n associated to Equation (0.1) is such that

$$\overline{X}_{\frac{k}{n}} = X_0 \prod_{i=1}^{k}\left(1+\frac{\sigma Z_i}{\sqrt{n}}\right)\,.$$

We proce this by recurrence (on k).

- Thus true for $k=0$.
- Suppose this is true in k.

$$\begin{aligned} \overline{X}_{\frac{k+1}{n}} &= \overline{X}_{\frac{k}{n}} + \sigma \overline{X}_{\frac{k}{n}}(W_{\frac{k+1}{n}} - W_{\frac{k}{n}}) \\ &= \overline{X}_{\frac{k}{n}}\left(1+\frac{\sigma Z_{k+1}}{\sqrt{n}}\right) \\ \text{(by recurrence)} \quad &= X_0 \prod_{i=1}^{k+1}\left(1+\frac{\sigma Z_i}{\sqrt{n}}\right)\,. \end{aligned}$$

(6) Show that the strong rate of convergence obtained in the course (in a theorem) is optimal.

The Theorem of the course tells us (if X_0 is constant)

$$\mathbb{E}(\sup_{0\leq k\leq n} |X_{\frac{k}{n}} - \overline{X}_{\frac{k}{n}}|^2) \leq \frac{C(1+X_0^2)}{n}$$

for some constant C. Here we have (by Questions 4,5)

$$\begin{aligned}
\mathbb{E}((X_1 - \overline{X}_1)^2) &= \mathbb{E}\left(\left(X_0 e^{\sigma(Z_1+\dots+Z_n)/\sqrt{n}-\sigma^2/2} - X_0 \prod_{k=1}^{n}\left(1+\frac{\sigma Z_k}{\sqrt{n}}\right)\right)^2\right) \\
\text{(by Question 3)} &= X_0^2 \frac{\sigma^2 e^{\sigma^2}}{2n}\left(\left(1 - \frac{1}{2}\left(\frac{2\sigma^2}{3n} + \frac{\sigma^4}{4n}\right) + O(1/n^2)\right)\right)^2 .
\end{aligned}$$

So the bound in $1/n$ is optimal.

(7) Write a code computing the Monte-Carlo approximation of the L^2 error for the above scheme, n fixed.

(8) Write a code producing a curve of the square of this L^2 error versus n (you can use a log-log scale and a linear regression to see the power of n). You should get something similar to Figure 0.1.

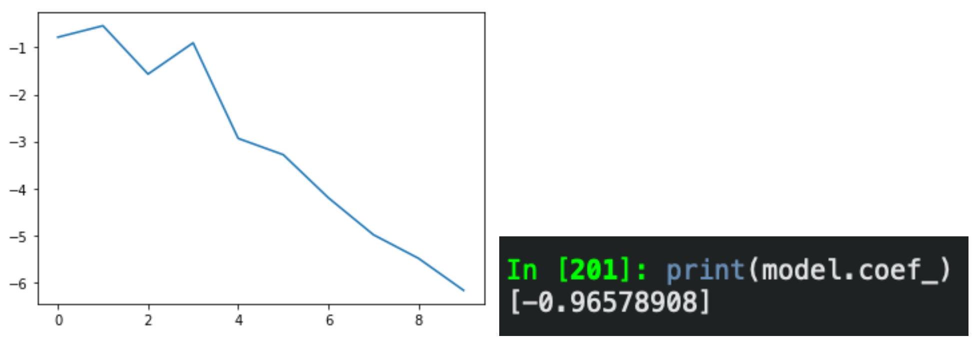

FIGURE 0.1. Rate of convergence in log-log scale.

PYTHON TIPS

Start by loading useful packages:

```
import numpy as np
import scipy.stats as sps
import matplotlib.pyplot as plt
import pandas as pd
from sklearn.linear_model import LinearRegression
```

Product/sum of terms in an array

```
np.prod(z1)
np.sum(z1)
```

Linear regression in dimension 1 (fit array `y` against array `x`) (get the slope)

```
model=LinearRegression()
model.fit(x.reshape((-1,1)),y)
print(model.coef_)
```